Quantum Field Theory: Exercise Session 5 21 December 2012

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Problem 1: Decay of a scalar particle

Consider the following Lagrangian, involving two real scalar fields Φ and φ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \mu\,\Phi\varphi\varphi. \tag{1}$$

The last term is an interaction term, with coupling constant μ, which allows the particle Φ to decay into two φ's.

(a) Write down the momentum space Feynman rules for this theory, and hence the Feynman diagram for the decay of Φ to lowest order in μ.

(b) Obtain the invariant matrix element $\mathcal{M}$, defined from the scattering matrix $S = 1+\mathrm{i}T$ by

$$\langle k_{\varphi_1} k_{\varphi_2}|\mathrm{i}T|k_\Phi\rangle = (2\pi)^4\,\delta^{(4)}(k_\Phi - k_{\varphi_1} - k_{\varphi_2})\,\mathrm{i}\mathcal{M}(k_\Phi \to k_{\varphi_1}, k_{\varphi_2})\,, \tag{2}$$

and given diagrammatically by

$$\mathrm{i}\mathcal{M} = \{\text{the sum of all connected amputated Feynman diagrams}\}\,. \tag{3}$$

(c) Compute the decay rate

$$\Gamma = \frac{1}{2M}\prod_{f=\varphi_1,\varphi_2}\int \tilde{\mathrm{d}}k_f\,|\mathcal{M}(k_\Phi \to k_{\varphi_1}, k_{\varphi_2})|^2\,(2\pi)^4\,\delta^{(4)}(k_\Phi - k_{\varphi_1} - k_{\varphi_2}) \tag{4}$$

and the life time $\tau = \Gamma^{-1}$ of the Φ particles in their rest frame to lowest order in μ. Hint: A change of variables $\delta(f(x))\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}y}\delta(y)\mathrm{d}y$ with $y = f(x)$ shows that

$$\frac{\delta\left(\sqrt{|\vec{k}|^2 + m_1^2} + \sqrt{|\vec{k}|^2 + m_2^2} - M\right)}{\sqrt{|\vec{k}|^2 + m_1^2}\sqrt{|\vec{k}|^2 + m_2^2}}|\vec{k}|^2\mathrm{d}|\vec{k}| = \frac{|\vec{k}|}{M}\delta(y)\mathrm{d}y \tag{5}$$

with $y = \sqrt{|\vec{k}|^2 + m_1^2} + \sqrt{|\vec{k}|^2 + m_2^2} - M$.

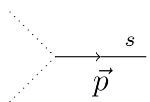
(d) What is the lower bound on M for a decay to be possible?

Problem 2: Decay of the charged pion

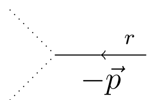
The negatively charged pion can be represented by a complex scalar field φ, the muon by a Dirac field ψ and the muon neutrino by a helicity-projected left-handed Dirac field $\frac{1}{2}(1-\gamma^5)\chi$. The pion decays almost exclusively into a muon and muon antineutrino, and this process can be described by the Lagrangian density

$$\begin{aligned} \mathcal{L} = \partial_\nu\varphi^*\partial^\nu\varphi - m_\pi^2|\varphi|^2 + \bar\psi(i\partial\!\!\!/ - m_\mu)\psi + \bar\chi i\partial\!\!\!/\chi \\ + \lambda\left[\partial_\nu\varphi\bar\psi\gamma^\nu(1-\gamma^5)\chi + \text{h.c.}\right], \end{aligned} \tag{6}$$

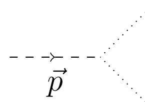
where λ is a coupling constant. The momentum space Fenyman rules for this theory read:

$\langle\vec{p}, s|\bar\psi \quad =$ 

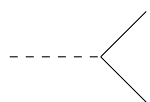
$= \bar{u}^s(p) \qquad$ fermion $\qquad (7)$

$\langle\vec{p}, r|\chi \quad =$ 

$= v^r(p) \qquad$ antifermion $\qquad (8)$

$\varphi|\vec{p}\rangle \quad =$ 

$= 1 \qquad (9)$


$= \lambda\, p_\nu\gamma^\nu(1-\gamma^5). \qquad (10)$

(a) Compute the lifetime of the pions.

Hint: When computing the modulus squared of the invariant matrix element, $|\mathcal{M}|^2$, you should sum over all possible final spin states using the completeness relations

$$\sum_{s=\pm} u_a^s(p)\bar{u}_b^s(p) = (p\!\!\!/ + m_\mu)_{ab} \qquad \text{where} \quad (p\!\!\!/ - m_\mu)u^s(p) = 0, \tag{11}$$

$$\sum_{r=\pm} v_a^r(p)\bar{v}_b^r(p) = p\!\!\!/_{ab} \qquad \text{where} \quad p\!\!\!/\, v^r(p) = 0, \tag{12}$$

and the gamma-trace identities

$$\mathrm{Tr}(\text{any odd number of } \gamma\text{'s}) = 0 \tag{13}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \tag{14}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0 \tag{15}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{16}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 4\mathrm{i}\epsilon^{\mu\nu\rho\sigma}. \tag{17}$$

(b) Why is the process $\pi^- \to e^- + \bar\nu_e$ suppressed?